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# Quantum field theory on quantum graphs and application to their conductance

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## Abstract

We construct a bosonic quantum field on a general quantum graph. Consistency of the construction leads to the calculation of the total scattering matrix of the graph. This matrix is equivalent to that already proposed using the generalized star-product approach. We give several examples and show how they generalize some of the scattering matrices computed in the mathematical or condensed matter physics literature. Then, we apply the construction for the calculation of the conductance of graphs, within a small distance approximation. The consistency of the approximation is proved by direct comparison with the exact calculation for the 'tadpole' graph.

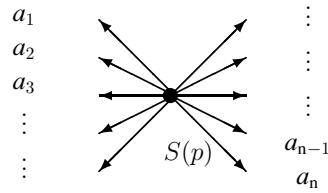
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## 1. Introduction

Quantum graphs have recently been the subject of intense studies, both at the mathematical level, see e.g. [1–11, 22] and references therein, and for condensed matter physics applications in wires, see e.g. [12–15] and references therein, or chaos [16]. These graphs appear to be a very good approximation for the modeling of quasi unidimensional systems, such as quantum or atomic wires (for a review see e.g. [17, 18]).

In this paper, we show how to construct quantum fields on a general graph, starting from the knowledge of the scattering matrix at each vertex of this graph. The construction relies on the RT-algebra formalism and gives a way to compute the total scattering matrix associated with the graph. This total scattering matrix is equivalent to that constructed using the generalized star-product framework [4, 6, 22]. Then, we apply the formalism to the explicit calculation of conductance for Tomonaga–Luttinger models for specific graphs, such as the tree graph, the loop, the tadpole and the triangle.

This paper consists of two parts. The first one (that contains sections 2, 3, 4 and 5) deals with the formal aspects of the construction. If one assumes the results of



**Figure 1.** Star graphs (the arrows indicate the orientation on the edges).

this part, one can directly read the second part (containing sections 6 and 7) that focuses on explicit calculations and examples.

More specifically, we summarize in section 2 known results [8–10] on quantum field theory on star graphs. In section 3, we show how to construct a bosonic quantum field on a graph consisting of two star graphs linked by a single line. The construction is essentially based on the determination of its total scattering matrix. In section 4, we generalize the approach to the case where several lines are tied between the two star graphs, and in section 5 we treat the general case of several star graphs linked by several lines. In section 6, we apply the previous results to the case of scale invariant scattering matrices. This allows us to recover results obtained both in the mathematical physics literature [6, 4, 22] and in condensed matter physics [13, 14]. Finally, using the techniques developed in [9, 11], we apply in section 7 our results to the calculation of conductance on graphs. The calculation is done in a short distance approximation. In the case of a tadpole graph, we compute the conductance exactly and show that the approximation is consistent with the exact calculation. An appendix is devoted to the proofs of the properties used in this paper.

## 2. Integrable field theory on star graphs

We summarize here the results developed in [8–10] for the construction of an integrable field theory on a star graph. The algebraic framework needed to define bosonic fields on a star graph is the RT-algebras [19, 20]. These algebras are a generalization of ZF-algebras, which themselves are a generalization of oscillator algebras. Indeed, if oscillator algebras are used to define free fields on, say, an infinite line, ZF-algebras are adapted to define interacting fields on this line, while RT-algebras take into account the introduction of a defect on this line.

### 2.1. RT-algebras

We present the RT-algebra for a star graph with  $n$  edges consisting of  $n$  infinite half-lines (the edges) originating from the same point (the vertex); see figure 1.

To each edge  $a = 1, \dots, n$ , one associates oscillator-like generators  $\{a_a(p), a_a^\dagger(p)\}$  that deal with the field propagating on the edge. They are gathered in row and line vectors:

$$A(p) = \begin{pmatrix} a_1(p) \\ a_2(p) \\ \vdots \\ a_n(p) \end{pmatrix} \quad \text{and} \quad A^\dagger(p) = (a_1^\dagger(p), a_2^\dagger(p), \dots, a_n^\dagger(p)). \quad (2.1)$$

It remains to give the (integrable) boundary condition at the vertex, i.e. the way the field connects between the different edges. This boundary condition is given by  $n^2$  generators  $s_{a_1 a_2}(p)$ , gathered in a matrix, the scattering matrix of the vertex:

$$S(p) = \begin{pmatrix} s_{11}(p) & s_{12}(p) & \cdots & s_{1n}(p) \\ s_{21}(p) & s_{22}(p) & \cdots & s_{2n}(p) \\ \vdots & & \ddots & \vdots \\ s_{n1}(p) & s_{n2}(p) & \cdots & s_{nn}(p) \end{pmatrix}. \quad (2.2)$$

The RT-algebra is the unital algebra generated by<sup>1</sup>  $\{\mathfrak{a}_a(p), \mathfrak{a}_b^\dagger(p), s_{ab}(p), a, b = 1, \dots, n, p \in \mathbb{R}\}$  submitted to the relations

$$\mathfrak{a}_{a_1}(p_1)\mathfrak{a}_{a_2}(p_2) - \mathfrak{a}_{a_2}(p_2)\mathfrak{a}_{a_1}(p_1) = 0, \quad (2.3)$$

$$\mathfrak{a}_{a_1}^\dagger(p_1)\mathfrak{a}_{a_2}^\dagger(p_2) - \mathfrak{a}_{a_2}^\dagger(p_2)\mathfrak{a}_{a_1}^\dagger(p_1) = 0, \quad (2.4)$$

$$\mathfrak{a}_{a_1}(p_1)\mathfrak{a}_{a_2}^\dagger(p_2) - \mathfrak{a}_{a_2}^\dagger(p_2)\mathfrak{a}_{a_1}(p_1) = 2\pi(\delta(p_1 - p_2)\delta_{a_1 a_2} + \delta(p_1 + p_2)s_{a_1 a_2}(p_1)) \quad (2.5)$$

and the boundary condition

$$A(p) = S(p)A(-p) \quad \text{and} \quad A^\dagger(p) = A^\dagger(-p)S(-p). \quad (2.6)$$

The RT-algebra admits an anti-automorphism (written for  $p \in \mathbb{R}$ ),

$$\mathfrak{a}_a(p) \rightarrow \mathfrak{a}_a^\dagger(p); \quad \mathfrak{a}_a^\dagger(p) \rightarrow \mathfrak{a}_a(p) \quad \text{and} \quad s_{a_1 a_2}(p) \rightarrow s_{a_2 a_1}(-p), \quad (2.7)$$

which is identified with the Hermitian conjugation.

There are two consistency relations coming from relation (2.6). For  $p \in \mathbb{R}$ , they read

$$S(p)S(-p) = \mathbb{I} \quad (2.8)$$

$$S^\dagger(p) = S(-p). \quad (2.9)$$

One recognizes in (2.9) the Hermitian analyticity for scattering matrix  $S(p)$ . Together with the consistency relation (2.8), it implies unitarity of the scattering matrix:

$$S(p)S^\dagger(p) = \mathbb{I}. \quad (2.10)$$

Below, we will decompose the scattering matrix into block submatrices:

$$S(p) = \left( \begin{array}{c|c} S_{11}(p) & S_{12}(p) \\ \hline S_{21}(p) & S_{22}(p) \end{array} \right). \quad (2.11)$$

Within this decomposition, the consistency relation (2.8) recasts into four equations:

$$S_{11}(p)S_{11}(-p) + S_{12}(p)S_{21}(-p) = \mathbb{I}; \quad S_{11}(p)S_{12}(-p) + S_{12}(p)S_{22}(-p) = 0 \quad (2.12)$$

$$S_{21}(p)S_{11}(-p) + S_{22}(p)S_{21}(-p) = 0; \quad S_{22}(p)S_{22}(-p) + S_{21}(p)S_{12}(-p) = \mathbb{I}. \quad (2.13)$$

<sup>1</sup> Strictly speaking, at the algebraic level, the RT-algebra can be defined for  $p \in \mathbb{C}$ . However, since  $p$  is physically associated with an impulsion, we restrict ourselves to real  $p$ 's.

### 2.2. The quantum field on the star graph

A massless bosonic field on the star graph is constructed from the RT-algebra generators as

$$\phi_a(x, t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{\sqrt{2|p|}} \{e^{-i(|p|t-px)} \mathbf{a}_a(p) + e^{i(|p|t-px)} \mathbf{a}_a^\dagger(p)\} \quad a = 1, 2, \dots, n. \quad (2.14)$$

In expression (2.14),  $x \geq 0$  is the distance on edge  $a$  on which the field propagates, with origin at the vertex. Using the relations (2.3)–(2.5) and (2.6), it can be shown that the field  $\phi$  has canonical equal time commutation on each edge,

$$\begin{aligned} [\phi_{a_1}(x_1, 0), \phi_{a_2}(x_2, 0)] &= 0 \quad \text{and} \quad [(\partial_t \phi_{a_1})(x_1, 0), \phi_{a_2}(x_2, 0)] = -i\delta_{a_1 a_2} \delta(x_1 - x_2), \\ x_1, x_2 > 0, \quad a_1, a_2 &= 1, \dots, n, \end{aligned} \quad (2.15)$$

and obeys the equation of motion,

$$(\partial_t^2 - \partial_x^2)\phi_a(x, t) = 0, \quad x > 0, \quad a = 1, 2, \dots, n, \quad (2.16)$$

and some boundary condition which depends on the form of  $S(p)$ . When the scattering matrix takes the form  $S(p) = -(B + ipC)^{-1}(B - ipC)$ , where  $B$  and  $C$  are real matrices such that  $BC^t = CB^t$ , this boundary condition reads

$$\sum_{b=1}^n (B_{ab}\phi_b(0, t) + C_{ba}(\partial_x \phi_b)(0, t)) = 0, \quad t \in \mathbb{R}, \quad a = 1, 2, \dots, n. \quad (2.17)$$

Equation (2.14) corresponds to a planar wave decomposition of the field  $\phi_a(x, t)$ . We will call  $\mathbf{a}_a(p)$  the mode (or the oscillator) on edge  $a$  (with momentum  $p$ ).

*Choice of the origin on each edge.* In the following, we will need to change the origin of coordinate on some edges. This amounts to changing the form of the scattering matrix. Indeed, from form (2.14), it is clear that a shift  $x \rightarrow x + d$  is equivalent to the transformation

$$\mathbf{a}_a(p) \rightarrow e^{ipd} \mathbf{a}_a(p) \quad \text{and} \quad \mathbf{a}_a^\dagger(p) \rightarrow e^{-ipd} \mathbf{a}_a^\dagger(p). \quad (2.18)$$

This transformation does not modify the relations (2.3) and (2.4), but it does affect the scattering matrix in (2.5). For general shifts of  $x \rightarrow x + d_a$ , on edge  $a = 1, \dots, n$ , the scattering matrix will be changed as follows:

$$S(p) \rightarrow W(p)S(p)W(p) \quad \text{with} \quad W(p) = \text{diag}(e^{ipd_1}, \dots, e^{ipd_n}). \quad (2.19)$$

Remark that since  $S(p)$  obeys the consistency and unitarity relations (2.8)–(2.10), the transformation (2.19) does not change the properties (2.8)–(2.10) of the scattering matrix.

In the same way, a change of orientation on the edges will correspond to a transformation:  $\mathbf{a}(p) \rightarrow \mathbf{a}(-p)$  in the boundary condition.

### 2.3. Star graphs as building blocks for quantum wires

In the following, we shall construct integrable quantum field theory on a general quantum wire. It should be clear that the star graphs can be considered as building blocks for such a general wire, in the same way the single defect on a line underlies the construction for several defects on the line [21]. In both cases, the above construction applies *locally*, around each vertex. The scattering matrices attached to each of the vertices will be called local. They are part of the quantum graph data. What remains to do is to connect these star graphs, i.e. identify the field on connecting edges between two star graphs. We will see that this physical identification is sufficient to determine the ‘internal’ modes (i.e., the generators  $\{\mathbf{a}_a(p), \mathbf{a}_a^\dagger(p)\}$  of a connecting edge  $a$  between two vertices) in terms of the ‘external’ modes. It also allows us to construct the global scattering matrix that relates the ‘external’ modes  $\{\mathbf{a}_b(p), \mathbf{a}_b^\dagger(p)\}$  (on

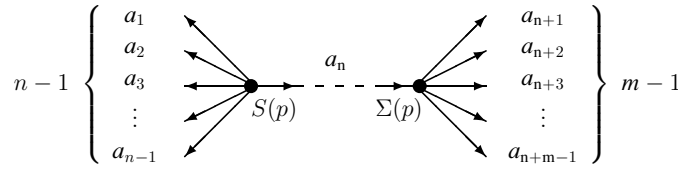


Figure 2. Simple gluing of two vertices (the arrows indicate the orientation of the edge).

external edges) through a relation of type (2.6). This natural identification (which leads to a purely algebraic calculation) appears to be equivalent to that introduced in [6] in analyzing the Schrödinger operator on graphs. In both cases, one needs to ‘glue’ star graphs together, either using a generalized star product [6] or through identification of the bosonic modes propagating on the connecting edge(s).

### 3. Simple gluing of two vertices

#### 3.1. General presentation

We consider two star graphs with  $n$  and  $m$  edges, respectively, that are linked by one edge. We want to construct the quantum field on this graph. The basic idea is that *locally* around each vertex, the bosonic field should be the same as that for the corresponding star graph. Then, one should connect the two constructions via the connecting edge, where the two fields should correspond. We call this procedure the ‘gluing’ of the two vertices. It is drawn in figure 2.

The local  $S$  matrices are denoted by

$$S(p) = \begin{pmatrix} s_{11}(p) & \dots & s_{1n}(p) \\ \vdots & & \vdots \\ s_{n1}(p) & \dots & s_{nn}(p) \end{pmatrix} \quad \text{and} \quad \Sigma(p) = \begin{pmatrix} \sigma_{11}(p) & \dots & \sigma_{1m}(p) \\ \vdots & & \vdots \\ \sigma_{m1}(p) & \dots & \sigma_{mm}(p) \end{pmatrix}. \tag{3.1}$$

The line that links the two vertices is denoted by  $n$  in  $S(p)$  and by  $1$  in  $\Sigma(p)$ . For each edge  $a \neq n$  the origin is chosen to be at the vertex to which the edge belongs. For edge  $n$ , the origin is chosen at the vertex described by  $S(p)$ , so that  $S(p)$  is the ‘true’ local scattering matrix of the vertex, while  $\Sigma(p)$  is related to the ‘true’ local scattering matrix  $\Sigma_0(p)$  by

$$\Sigma(p) = W(p)\Sigma_0(p)W(p) \quad \text{with} \quad W(p) = \text{diag}(e^{ipd_n}, 1, 1, \dots, 1), \tag{3.2}$$

where  $d_n$  is the distance between the two vertices (measured on edge  $n$ ).

The boundary conditions on each vertex are local, and hence take the form

$$\widehat{A}(p) = S(p)\widehat{A}(-p) \quad \text{and} \quad \widehat{B}(p) = \Sigma(p)\widehat{B}(-p), \tag{3.3}$$

where

$$\widehat{A}(p) = \begin{pmatrix} a_1(p) \\ a_2(p) \\ \vdots \\ a_n(p) \end{pmatrix} \quad \text{and} \quad \widehat{B}(p) = \begin{pmatrix} a_{n+1}(-p) \\ a_{n+1}(p) \\ \vdots \\ a_{n+m-1}(p) \end{pmatrix}. \tag{3.4}$$

Since the mode  $\mathfrak{a}_n(p)$  is common to  $A(p)$  and  $B(p)$ , one can eliminate it from the system. In other words, the field on the ‘inner line’ of the graph is constructed from the modes on the outer lines. In order to do this calculation, we single out  $\mathfrak{a}_n(p)$ ,

$$\widehat{A}(p) = (A(p)\mathfrak{a}_n(p)) \quad \text{and} \quad \widehat{B}(p) = \begin{pmatrix} \mathfrak{a}_n(-p) \\ B(p) \end{pmatrix}, \quad (3.5)$$

where we have introduced

$$A(p) = \begin{pmatrix} \mathfrak{a}_1(p) \\ \vdots \\ \mathfrak{a}_{n-1}(p) \end{pmatrix} \quad \text{and} \quad B(p) = \begin{pmatrix} \mathfrak{a}_{n+1}(p) \\ \vdots \\ \mathfrak{a}_{n+m-1}(p) \end{pmatrix}. \quad (3.6)$$

We apply the same decomposition to matrices  $S(p)$  and  $\Sigma(p)$ :

$$\begin{aligned} S_{11}(p) &= \begin{pmatrix} s_{11}(p) & \dots & s_{1,n-1}(p) \\ \vdots & & \vdots \\ s_{n-1,1}(p) & \dots & s_{n-1,n-1}(p) \end{pmatrix}; & \Sigma_{22}(p) &= \begin{pmatrix} \sigma_{22}(p) & \dots & \sigma_{2m}(p) \\ \vdots & & \vdots \\ \sigma_{m2}(p) & \dots & \sigma_{mm}(p) \end{pmatrix} \\ S_{21}(p) &= (s_{n1}(p), \dots, s_{n,n-1}(p)); & \Sigma_{12}(p) &= (\sigma_{12}(p), \dots, \sigma_{1m}(p)) \\ S_{12}(p) &= \begin{pmatrix} s_{1n}(p) \\ \vdots \\ s_{n-1,n}(p) \end{pmatrix}; & \Sigma_{21}(p) &= \begin{pmatrix} \sigma_{21}(p) \\ \vdots \\ \sigma_{m1}(p) \end{pmatrix} \end{aligned} \quad (3.7)$$

so that

$$S(p) = \begin{pmatrix} S_{11}(p) & S_{12}(p) \\ S_{21}(p) & s_{nn}(p) \end{pmatrix} \quad \Sigma(p) = \begin{pmatrix} \sigma_{11}(p) & \Sigma_{12}(p) \\ \Sigma_{21}(p) & \Sigma_{22}(p) \end{pmatrix}. \quad (3.8)$$

With these splittings, the boundary conditions (3.3) recast as

$$A(p) = S_{11}(p)A(-p) + S_{12}(p)\mathfrak{a}_n(-p) \quad (3.9)$$

$$\mathfrak{a}_n(p) = S_{21}(p)A(-p) + s_{nn}(p)\mathfrak{a}_n(-p) \quad (3.10)$$

$$B(p) = \Sigma_{22}(p)B(-p) + \Sigma_{21}(p)\mathfrak{a}_n(p) \quad (3.11)$$

$$\mathfrak{a}_n(-p) = \Sigma_{12}(p)B(-p) + \sigma_{11}(p)\mathfrak{a}_n(p). \quad (3.12)$$

Equations (3.10) and (3.12) allow us to express  $\mathfrak{a}_n(p)$  in terms of  $A(p)$  and  $B(p)$

$$\mathfrak{a}_n(p) = \frac{1}{1 - \sigma_{11}(p)s_{nn}(p)} (S_{21}(p)A(-p) + s_{nn}(p)\Sigma_{12}(p)B(-p)) \quad (3.13)$$

together with a consistency relation

$$\begin{aligned} S_{21}(p)A(-p) + s_{nn}(p)\Sigma_{12}(p)B(-p) \\ = \frac{1 - \sigma_{11}(p)s_{nn}(p)}{1 - \sigma_{11}(-p)s_{nn}(-p)} (\sigma_{11}(-p)S_{21}(-p)A(p) + \Sigma_{12}(-p)B(p)). \end{aligned} \quad (3.14)$$

This consistency relation is automatically satisfied if  $S(p)$  and  $\Sigma(p)$  obey the consistency relation (2.8); see proof in appendix A.1 for a more general case. Then, defining

$$\mathcal{A}(p) = \begin{pmatrix} A(p) \\ B(p) \end{pmatrix} \quad (3.15)$$

we recast the two remaining relations as

$$\mathcal{A}(p) = S_{\text{tot}}(p)\mathcal{A}(-p) \quad (3.16)$$

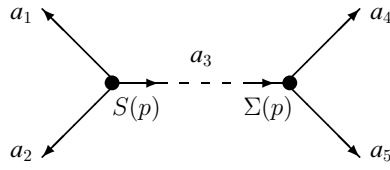


Figure 3. The tree graph.

with

$$S_{\text{tot}}(p) = \begin{pmatrix} S_{11}(p) + \frac{\sigma_{11}(p)S_{12}(p)S_{21}(p)}{1 - \sigma_{11}(p)s_{nn}(p)} & \frac{S_{12}(p)\Sigma_{12}(p)}{1 - \sigma_{11}(p)s_{nn}(p)} \\ \frac{\Sigma_{21}(p)S_{21}(p)}{1 - \sigma_{11}(p)s_{nn}(p)} & \Sigma_{22}(p) + \frac{s_{nn}(p)\Sigma_{21}(p)\Sigma_{12}(p)}{1 - \sigma_{11}(p)s_{nn}(p)} \end{pmatrix}. \quad (3.17)$$

One can show that if  $S(p)$  and  $\Sigma(p)$  obey the consistency relation (2.8), then so does  $S_{\text{tot}}(p)$ . In the same way, the unitarity relation (2.10) for matrices  $S(p)$  and  $\Sigma(p)$  implies unitarity for the matrix  $S_{\text{tot}}(p)$ .

The bosonic quantum field  $\phi_a(x, t)$  keeps the form (2.14). Since the total scattering matrix obeys relations (2.9) and (2.8), the field  $\phi_a(x, t)$  on external edges ( $a \neq n$ ) still obeys relations (2.15) and (2.16). However, on edge  $n$ , one has to replace the generators  $\{a_n(p), a_n^\dagger(p)\}$  by their expression (3.13), and it is not ensured that  $\phi_n(x, t)$  is canonical.

Remark that the scattering matrix can be rewritten as

$$S_{\text{tot}}(p) = \begin{pmatrix} S_{11}(p) & 0 \\ 0 & \Sigma_{22}(p) \end{pmatrix} + \frac{1}{1 - \sigma_{11}(p)s_{nn}(p)} \times \begin{pmatrix} \sigma_{11}(p)S_{12}(p)S_{21}(p) & S_{12}(p)\Sigma_{12}(p) \\ \Sigma_{21}(p)S_{21}(p) & s_{nn}(p)\Sigma_{21}(p)\Sigma_{12}(p) \end{pmatrix}. \quad (3.18)$$

The first term in (3.18) corresponds to the scattering matrix of the two local vertices without interaction (i.e., when edge  $n$  is removed), while the second term is the ‘perturbation’ due to the link through  $a_n(p)$ . Let us stress that only  $S_{\text{tot}}(p)$  is unitary.

**Remark 3.1.** In the limit of a vanishing distance between the vertices,  $d_n \rightarrow 0$ , the gluing of scattering matrices can be viewed as a recursive process to build higher dimensional scattering matrices, starting from low-dimensional ones. The process ensures unitarity of the final matrix when the original ones are.

### 3.2. Example 1: the ‘tree graph’

As an example, we consider the gluing along one edge of the two vertices with three edges. In this way, we construct the scattering matrix for a vertex with four edges that we call a ‘tree graph’ for obvious particle physics reasons; see figure 3. This gluing is the simplest example of the recursive process mentioned in remark 3.1.

The decomposition of the local  $S$  matrices reads

$$S(p) = \left( \begin{array}{cc|c} s_{11}(p) & s_{12}(p) & s_{13}(p) \\ s_{21}(p) & s_{22}(p) & s_{23}(p) \\ s_{31}(p) & s_{32}(p) & s_{33}(p) \end{array} \right) \quad \text{and} \quad \Sigma(p) = \left( \begin{array}{cc|c} \sigma_{11}(p) & \sigma_{12}(p) & \sigma_{13}(p) \\ \sigma_{21}(p) & \sigma_{22}(p) & \sigma_{23}(p) \\ \sigma_{31}(p) & \sigma_{32}(p) & \sigma_{33}(p) \end{array} \right), \quad (3.19)$$



where the lines indicate the block decomposition of the matrices. The local boundary conditions have the form

$$\begin{pmatrix} \mathbf{a}_1(p) \\ \mathbf{a}_2(p) \\ \mathbf{a}_3(p) \end{pmatrix} = S(p) \begin{pmatrix} \mathbf{a}_1(-p) \\ \mathbf{a}_2(-p) \\ \mathbf{a}_3(-p) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{a}_3(-p) \\ \mathbf{a}_4(p) \\ \mathbf{a}_5(p) \end{pmatrix} = \Sigma(p) \begin{pmatrix} \mathbf{a}_3(p) \\ \mathbf{a}_4(-p) \\ \mathbf{a}_5(-p) \end{pmatrix}. \quad (3.20)$$

We will focus on identical local vertices. This does *not* mean that local matrices  $S(p)$  and  $\Sigma(p)$  are identical, because of the different labeling and orientation of the edges on the total graph, and also because of the choice of the origin on the connecting edge. The local scattering matrices rather obey

$$\begin{aligned} \Sigma(p) &= W(p)PS(-p)P^{-1}W(p) \quad \text{with} \\ P &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad W(p) = \text{diag}(e^{ipd}, 1, 1), \end{aligned} \quad (3.21)$$

where  $d$  is the distance between the two vertices. In (3.21),  $P$  rotates the  $S$ -matrix according to the labeling of the edges, while  $W(p)$  implements the shift of the origin, according to (2.19). It leads to

$$\Sigma(p) = \left( \begin{array}{c|cc} s_{33}(-p)e^{2ipd} & s_{31}(-p)e^{ipd} & s_{32}(-p)e^{ipd} \\ \hline s_{13}(-p)e^{ipd} & s_{11}(-p) & s_{12}(-p) \\ s_{23}(-p)e^{ipd} & s_{21}(-p) & s_{22}(-p) \end{array} \right). \quad (3.22)$$

Using this expression and the consistency relation, one rewrites (3.17) as

$$S_{\text{tot}}(p) = \begin{pmatrix} S_{11}(p) & 0 \\ 0 & S_{11}(-p) \end{pmatrix} + \frac{e^{2ipd}}{\mathcal{N}(p)} \begin{pmatrix} s_{33}(-p)M(p, p) & e^{-ipd}M(p, -p) \\ e^{-ipd}M(-p, p) & s_{33}(p)M(-p, -p) \end{pmatrix} \quad (3.23)$$

$$\mathcal{N}(p) = 1 - e^{2ipd}s_{33}(p)s_{33}(-p),$$

where we have introduced the submatrix

$$M(p, q) = S_{12}(p) \cdot S_{21}(q) = \begin{pmatrix} s_{13}(p)s_{31}(q) & s_{13}(p)s_{32}(q) \\ s_{23}(p)s_{31}(q) & s_{23}(p)s_{32}(q) \end{pmatrix}. \quad (3.24)$$

The boundary condition for the total tree graph reads

$$\begin{pmatrix} \mathbf{a}_1(p) \\ \mathbf{a}_2(p) \\ \mathbf{a}_4(p) \\ \mathbf{a}_5(p) \end{pmatrix} = S_{\text{tot}}(p) \begin{pmatrix} \mathbf{a}_1(-p) \\ \mathbf{a}_2(-p) \\ \mathbf{a}_4(-p) \\ \mathbf{a}_5(-p) \end{pmatrix}, \quad (3.25)$$

and the ‘inner mode’  $\mathbf{a}_3(p)$  is expressed in terms of the ‘outer modes’ as

$$\begin{aligned} \mathbf{a}_3(p) &= \frac{1}{\mathcal{N}(p)} \{s_{31}(p)\mathbf{a}_1(-p) + s_{32}(p)\mathbf{a}_2(-p) \\ &\quad + e^{ipd}s_{33}(p)(s_{31}(-p)\mathbf{a}_4(-p) + s_{32}(-p)\mathbf{a}_5(-p))\}. \end{aligned} \quad (3.26)$$

#### 4. General gluing of two vertices

We now turn to the case of two vertices linked by  $\tau$  lines, as shown in figure 4.

Following the same techniques as in section 3, it is clear that the construction of the quantum field on the graph is equivalent to the determination of the total scattering matrix for the complete graph.

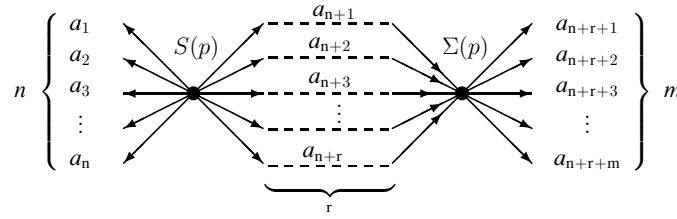


Figure 4. General gluing of two vertices.

4.1. General case

As in the previous case, we divide the local  $S$  matrices according to the  $\tau$  lines that are common to the two vertices,

$$S(p) = \begin{pmatrix} S_{11}(p) & S_{12}(p) \\ S_{21}(p) & S_{22}(p) \end{pmatrix}; \quad \Sigma(p) = \begin{pmatrix} \Sigma_{11}(p) & \Sigma_{12}(p) \\ \Sigma_{21}(p) & \Sigma_{22}(p) \end{pmatrix}, \quad (4.1)$$

where the block submatrices have sizes  $n \times n, n \times \tau, \tau \times n, \tau \times \tau$  in  $S(p)$ :

$$\begin{aligned} S_{11}(p) &= \begin{pmatrix} s_{11}(p) & \dots & s_{1n}(p) \\ \vdots & & \vdots \\ s_{n1}(p) & \dots & s_{nn}(p) \end{pmatrix} & S_{12}(p) &= \begin{pmatrix} s_{1,n+1}(p) & \dots & s_{1,n+\tau}(p) \\ \vdots & & \vdots \\ s_{n,n+1}(p) & \dots & s_{n,n+\tau}(p) \end{pmatrix} \\ S_{21}(p) &= \begin{pmatrix} s_{n+1,1}(p) & \dots & s_{n+1,n}(p) \\ \vdots & & \vdots \\ s_{n+\tau,1}(p) & \dots & s_{n+\tau,n}(p) \end{pmatrix} & S_{22}(p) &= \begin{pmatrix} s_{n+1,n+1}(p) & \dots & s_{n+1,n+\tau}(p) \\ \vdots & & \vdots \\ s_{n+\tau,n+1}(p) & \dots & s_{n+\tau,n+\tau}(p) \end{pmatrix}, \end{aligned} \quad (4.2)$$

and sizes  $m \times m, m \times \tau, \tau \times m, \tau \times \tau$  in  $\Sigma(p)$ :

$$\begin{aligned} \Sigma_{11}(p) &= \begin{pmatrix} \sigma_{11}(p) & \dots & \sigma_{1\tau}(p) \\ \vdots & & \vdots \\ \sigma_{\tau 1}(p) & \dots & \sigma_{\tau\tau}(p) \end{pmatrix} & \Sigma_{12}(p) &= \begin{pmatrix} \sigma_{1,\tau+1}(p) & \dots & \sigma_{1,m+\tau}(p) \\ \vdots & & \vdots \\ \sigma_{\tau,\tau+1}(p) & \dots & \sigma_{\tau,m+\tau}(p) \end{pmatrix} \\ \Sigma_{21}(p) &= \begin{pmatrix} \sigma_{\tau+1,1}(p) & \dots & \sigma_{\tau+1,\tau}(p) \\ \vdots & & \vdots \\ \sigma_{m+\tau,1}(p) & \dots & \sigma_{m+\tau,\tau}(p) \end{pmatrix} & \Sigma_{22}(p) &= \begin{pmatrix} \sigma_{\tau+1,\tau+1}(p) & \dots & \sigma_{\tau+1,m+\tau}(p) \\ \vdots & & \vdots \\ \sigma_{m+\tau,\tau+1}(p) & \dots & \sigma_{m+\tau,m+\tau}(p) \end{pmatrix}. \end{aligned} \quad (4.3)$$

Modes  $\widehat{A}(p)$  and  $\widehat{B}(p)$  on each local vertex are decomposed accordingly,

$$\widehat{A}(p) = \begin{pmatrix} A_1(p) \\ A_2(p) \end{pmatrix}; \quad \widehat{B}(p) = \begin{pmatrix} A_2(-p) \\ A_3(p) \end{pmatrix}, \quad (4.4)$$

where

$$A_1(p) = \begin{pmatrix} \mathbf{a}_1(p) \\ \vdots \\ \mathbf{a}_n(p) \end{pmatrix}; \quad A_2(p) = \begin{pmatrix} \mathbf{a}_{n+1}(p) \\ \vdots \\ \mathbf{a}_{n+\tau}(p) \end{pmatrix}; \quad A_3(p) = \begin{pmatrix} \mathbf{a}_{n+\tau+1}(p) \\ \vdots \\ \mathbf{a}_{n+m+\tau}(p) \end{pmatrix}. \quad (4.5)$$

The calculation follows the same lines as in section 3 and we get

$$\mathcal{A}(p) = S_{\text{tot}}(p)\mathcal{A}(-p) \quad \text{with} \quad \mathcal{A}(p) = \begin{pmatrix} A_1(p) \\ A_3(p) \end{pmatrix} \quad (4.6)$$

and

$$S_{\text{tot}}(p) = \begin{pmatrix} S_{11}(p) + S_{12}(p)D(p)^{-1}\Sigma_{11}(p)S_{21}(p) & S_{12}(p)D(p)^{-1}\Sigma_{12}(p) \\ \Sigma_{21}(p)\tilde{D}(p)^{-1}S_{21}(p) & \Sigma_{22}(p) + \Sigma_{21}(p)\tilde{D}(p)^{-1}S_{22}(p)\Sigma_{12}(p) \end{pmatrix}, \quad (4.7)$$

where

$$D(p) = \mathbb{I}_\tau - \Sigma_{11}(p)S_{22}(p) \quad \text{and} \quad \tilde{D}(p) = \mathbb{I}_\tau - S_{22}(p)\Sigma_{11}(p) \quad (4.8)$$

is now an  $\tau \times \tau$  matrix supposed to be invertible (which is true for generic values of  $d$ ; the distance between the two vertices).

One checks easily that the formulae (4.7) are identical to that given by the star-product approach; see, e.g., formula (33) in [4] and formula (3.4) in [6]. Matrices  $D(p)^{-1}$  and  $\tilde{D}(p)^{-1}$  in this paper correspond (through a series expansion) to matrices  $K_1$  and  $K_2$  there, and the assumption of invertibility of  $D(p)$  and  $\tilde{D}(p)$  is the compatibility condition assumed in [4, 6]. In the language of [4, 6], we have made the generalized star product  $S(p) *_{W(p)} \Sigma(p)$ .

The present approach also allows us to reconstruct the modes in between the two vertices from those outside:

$$A_2(p) = \tilde{D}(p)^{-1}(S_{21}(p)A_1(-p) + S_{22}(p)\Sigma_{12}(p)A_3(-p)). \quad (4.9)$$

As in section 3, there is an additional consistency relation that is automatically satisfied if  $S(p)$  and  $\Sigma(p)$  obey the consistency relation (2.8). The proof is given in appendix A.1. In this case,  $S_{\text{tot}}(p)$  also obeys this relation. The same is true for the unitarity relation. When we take  $\tau = 1$ , we recover the case of section 3.

#### 4.2. Case of identical vertices

To simplify the expression of  $S_{\text{tot}}(p)$  given above, we now focus on identical vertices. As already mentioned, due to the different labeling of the edges, the orientation of the edges and the choice of the origin on the connecting edge, the local scattering matrices are not identical, but rather obey

$$S(p) = \begin{pmatrix} S_{11}(p) & S_{12}(p) \\ S_{21}(p) & S_{22}(p) \end{pmatrix} \quad \text{and} \quad \Sigma(p) = \begin{pmatrix} e^{2ipd}S_{22}(-p) & e^{ipd}S_{21}(-p) \\ e^{ipd}S_{12}(-p) & S_{11}(-p) \end{pmatrix}, \quad (4.10)$$

where we assumed that the distance between the two vertices is  $d$ , whatever the connecting edge on which it is measured. Since the vertices are identical, one has  $n = m$ , and  $S_{11}(p)$  is a  $n \times n$  matrix, while  $S_{22}(p)$  is  $\tau \times \tau$ .

Then, using consistency relations (2.12)–(2.13), which in particular implies

$$S_{12}(p)(\mathbb{I}_n - e^{2ipd}S_{22}(-p)S_{22}(p)) = (\mathbb{I}_n - e^{2ipd}S_{11}(p)S_{11}(-p))S_{12}(p), \quad (4.11)$$

one can rewrite (4.7) as

$$S_{\text{tot}}(p) = \begin{pmatrix} (1 - e^{2ipd})D_1(p)^{-1}S_{11}(p) & e^{ipd}D_1(p)^{-1}(\mathbb{I}_\tau - S_{11}(p)S_{11}(-p)) \\ e^{ipd}\tilde{D}_1(p)^{-1}(\mathbb{I}_\tau - S_{11}(-p)S_{11}(p)) & (1 - e^{2ipd})\tilde{D}_1(p)^{-1}S_{11}(-p) \end{pmatrix}, \quad (4.12)$$

where

$$D_1(p) = \mathbb{I}_\tau - e^{2ipd}S_{11}(p)S_{11}(-p) \quad \text{and} \quad \tilde{D}_1(p) = \mathbb{I}_\tau - e^{2ipd}S_{11}(-p)S_{11}(p). \quad (4.13)$$

Remark that the total scattering matrix is built on the block submatrix  $S_{11}(p)$  solely.

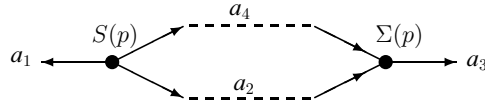


Figure 5. The loop.

The modes on the inner edges are given by

$$A_2(p) = (\mathbb{I}_n - e^{2ipd} S_{22}(-p)S_{22}(p))^{-1} (S_{21}(p)A_1(-p) + e^{ipd} S_{22}(p)S_{21}(-p)A_3(-p)). \tag{4.14}$$

4.3. Approximation for small distance

Taking the limit  $d \rightarrow 0$ , one gets the trivial scattering matrix

$$S_{\text{tot}}(p)|_{d=0} = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \tag{4.15}$$

corresponding to  $n$  non-connected infinite lines.

Thus, one could think of an expansion of  $S_{\text{tot}}(p)$  in terms of distance  $d$  to get new scattering matrices. However, the physical data, such as the conductance (see section 7), rely heavily on the pole structure of the scattering matrix. Hence, before we perform an approximation of (4.12) for small  $d$ , we rewrite it as (for  $t = \tan(pd/2)$ )

$$S_{\text{tot}}(p) = \begin{pmatrix} S'_{11}(p) & S'_{12}(p) \\ S'_{21}(p) & S'_{22}(p) \end{pmatrix} \tag{4.16}$$

$$S'_{11}(p) = -4it\mathfrak{D}(p)^{-1}S_{11}(p); \quad S'_{12}(p) = (1 + t^2)\mathfrak{D}(p)^{-1}(\mathbb{I} - S_{11}(p)S_{11}(-p)) \tag{4.17}$$

$$\mathfrak{D}(p) = (1 - it)^2\mathbb{I} - (1 + it)^2S_{11}(p)S_{11}(-p) \tag{4.18}$$

$$S'_{21}(p) = (1 + t^2)\tilde{\mathfrak{D}}(p)^{-1}(\mathbb{I} - S_{11}(-p)S_{11}(p)); \quad S'_{22}(p) = -4it\tilde{\mathfrak{D}}(p)^{-1}S_{11}(-p) \tag{4.19}$$

$$\tilde{\mathfrak{D}}(p) = (1 - it)^2\mathbb{I} + (1 + it)^2S_{11}(-p)S_{11}(p). \tag{4.20}$$

Then, the approximation is done using the expansion  $t \sim pd/2$ , but keeping the possible fractions entering the formulae above. We detail in section 6 this expansion for some examples: it will show how the pole structure is (partially) preserved in such an expansion. In section 7, we apply this expansion to the calculation of the conductance. In particular, we will show its consistency by comparing in one example this approximation to the full calculation of the conductance.

4.4. Example 2: the loop graph

In this case, one considers two vertices with three edges each, two of them being glued together. The local  $S$ -matrices are  $3 \times 3$ , and the total  $S$ -matrix (after gluing) is  $2 \times 2$ . In the notation of the previous section, we have  $n = m = 3$  and  $\tau = 2$ .

The decomposition of the local  $S$  matrices reads

$$S(p) = \left( \begin{array}{c|cc} s_{11}(p) & s_{12}(p) & s_{13}(p) \\ s_{21}(p) & s_{22}(p) & s_{23}(p) \\ s_{31}(p) & s_{32}(p) & s_{33}(p) \end{array} \right) \quad \text{and} \quad \Sigma(p) = \left( \begin{array}{c|cc} \sigma_{11}(p) & \sigma_{12}(p) & \sigma_{13}(p) \\ \sigma_{21}(p) & \sigma_{22}(p) & \sigma_{23}(p) \\ \sigma_{31}(p) & \sigma_{32}(p) & \sigma_{33}(p) \end{array} \right). \tag{4.21}$$

Again, the lines drawn within the matrices indicate the block submatrices we consider. The local boundary conditions have the form

$$\begin{pmatrix} \mathbf{a}_1(p) \\ \mathbf{a}_2(p) \\ \mathbf{a}_4(p) \end{pmatrix} = S(p) \begin{pmatrix} \mathbf{a}_1(-p) \\ \mathbf{a}_2(-p) \\ \mathbf{a}_4(-p) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{a}_2(-p) \\ \mathbf{a}_4(-p) \\ \mathbf{a}_3(p) \end{pmatrix} = \Sigma(p) \begin{pmatrix} \mathbf{a}_2(p) \\ \mathbf{a}_4(p) \\ \mathbf{a}_3(-p) \end{pmatrix}. \quad (4.22)$$

We will focus on identical vertices:

$$\begin{aligned} \Sigma(p) &= W(p)P^{-1}S(-p)PW(p) \quad \text{with} \\ P &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad W(p) = \text{diag}(e^{ipd_2}, e^{ipd_4}, 1), \end{aligned} \quad (4.23)$$

where  $d_a$  is the distance between the two vertices, measured on edges  $a = 2, 4$ . It leads to

$$\Sigma(p) = \left( \begin{array}{cc|c} s_{22}(-p) e^{2ipd_2} & s_{23}(-p) e^{ip(d_2+d_4)} & s_{21}(-p) e^{ipd_2} \\ s_{32}(-p) e^{ip(d_2+d_4)} & s_{33}(-p) e^{2ipd_4} & s_{31}(-p) e^{ipd_4} \\ \hline s_{12}(-p) e^{ipd_2} & s_{13}(-p) e^{ipd_4} & s_{11}(-p) \end{array} \right). \quad (4.24)$$

To get simple expressions, we suppose that  $d_2 = d_4 = d/2$ , where  $d$  is the total length of the loop. The total scattering matrix takes the form (with  $t = \tan(dp/2)$ ):

$$S_{\text{tot}}(p) = \frac{1}{\mathcal{N}(p)} \begin{pmatrix} -4its_{11}(p) & (1+t^2)(1-s_{11}(p)s_{11}(-p)) \\ (1+t^2)(1-s_{11}(p)s_{11}(-p)) & -4its_{11}(-p) \end{pmatrix} \quad (4.25)$$

$$\mathcal{N}(p) = (1-it)^2 - (1+it)^2 s_{11}(-p)s_{11}(p).$$

It corresponds for the total graph to a boundary condition

$$\begin{pmatrix} \mathbf{a}_1(p) \\ \mathbf{a}_3(p) \end{pmatrix} = S_{\text{tot}}(p) \begin{pmatrix} \mathbf{a}_1(-p) \\ \mathbf{a}_3(-p) \end{pmatrix}. \quad (4.26)$$

The inner modes read

$$\mathbf{a}_2(p) = -\frac{s_{21}(p)}{\mathcal{N}(p)} (\mathbf{a}_1(-p) + e^{ipd} s_{11}(-p) \mathbf{a}_3(-p)) \quad (4.27)$$

$$\mathbf{a}_4(p) = -\frac{s_{31}(p)}{\mathcal{N}(p)} (\mathbf{a}_1(-p) + e^{ipd} s_{11}(-p) \mathbf{a}_3(-p)). \quad (4.28)$$

*4.4.1. Expansion for short distances.* The general formula (4.25) simplifies to

$$\begin{aligned} S_{\text{tot}}(p) &\sim \frac{1}{\mathcal{N}_0(p)} \begin{pmatrix} -2ipds_{11}(p) & 1-s_{11}(p)s_{11}(-p) \\ 1-s_{11}(p)s_{11}(-p) & -2ipds_{11}(-p) \end{pmatrix} \\ \mathcal{N}_0(p) &= \left(1 - i\frac{dp}{2}\right)^2 - \left(1 + i\frac{dp}{2}\right)^2 s_{11}(-p)s_{11}(p), \end{aligned} \quad (4.29)$$

where now  $s_{11}(p)$  is a scalar function. If one assumes furthermore that  $s_{11}(p)$  is a constant (see section 6), the expansion leads to a total scattering matrix with two simple poles  $\frac{2i}{d} \frac{s_{11} \pm 1}{s_{11} \mp 1}$ .

#### 4.5. Example 3: the tadpole graph

The tadpole is constructed as a special case of the loop, where one of the vertices is fully transmitting between two edges, and purely reflexive in the third edge (with coefficient 1). In

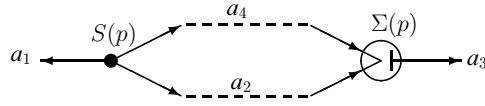


Figure 6. The tadpole (plus a half-line).

this way, we get a system with a tadpole and a decoupled half-line, as depicted in figure 6. The local scattering matrices of this graph are not identical: the first one has the general form

$$S(p) = \left( \begin{array}{ccc|ccc} s_{11}(p) & s_{12}(p) & s_{13}(p) & & & \\ s_{21}(p) & s_{22}(p) & s_{23}(p) & & & \\ s_{31}(p) & s_{32}(p) & s_{33}(p) & & & \end{array} \right) = \begin{pmatrix} S_{11}(p) & S_{12}(p) \\ S_{21}(p) & S_{22}(p) \end{pmatrix} \quad (4.30)$$

while the one associated with the purely reflexive vertex reads

$$\Sigma(p) = \left( \begin{array}{cc|c} 0 & e^{ip(d_2+d_4)} & 0 \\ e^{ip(d_2+d_4)} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) = \begin{pmatrix} \Sigma_{11}(p) & \Sigma_{12}(p) \\ \Sigma_{21}(p) & \Sigma_{22}(p) \end{pmatrix}, \quad (4.31)$$

where  $d_2$  and  $d_4$  are the distances between the two vertices, measured on edges 2 and 4 respectively, so that the length of the loop in the tadpole is  $\ell = d_2 + d_4$ .

From formulae (4.7)–(4.9), we get

$$\begin{aligned} S_{\text{tot}}(p) &= \begin{pmatrix} R(p) & 0 \\ 0 & 1 \end{pmatrix} \\ R(p) &= s_{11}(p) + \frac{e^{ip\ell}}{N(p)} (R_0(p) + e^{ip\ell} R_1(p)) \\ R_0(p) &= s_{12}(p)s_{31}(p) + s_{13}(p)s_{21}(p) \\ R_1(p) &= s_{12}(p)(s_{33}(p)s_{21}(p) - s_{23}(p)s_{31}(p)) + s_{13}(p)(s_{22}(p)s_{31}(p) - s_{32}(p)s_{21}(p)) \\ N(p) &= (1 - e^{ip\ell}s_{23}(p))(1 - e^{ip\ell}s_{32}(p)) - e^{2ip\ell}s_{22}(p)s_{33}(p). \end{aligned} \quad (4.32)$$

The modes in between the two vertices are reconstructed from those outside:

$$\begin{aligned} \begin{pmatrix} \mathbf{a}_2(p) \\ \mathbf{a}_4(p) \end{pmatrix} &= \tilde{D}(p)^{-1} S_{21}(p) \mathbf{a}_1(-p) \\ &= \begin{pmatrix} s_{21}(p) + (s_{22}(p)s_{31}(p) - s_{32}(p)s_{21}(p)) e^{ip\ell} \\ s_{31}(p) + (s_{33}(p)s_{21}(p) - s_{23}(p)s_{31}(p)) e^{ip\ell} \end{pmatrix} \frac{\mathbf{a}_1(-p)}{N(p)}. \end{aligned} \quad (4.33)$$

As expected, the mode  $\mathbf{a}_3(p)$  on the purely reflexive half-line decouples, and the mode(s) on the loop of the tadpole depends solely on  $\mathbf{a}_1(p)$ , the mode on the outer line of the tadpole. This mode obeys a reflection boundary condition.

Again, one can perform an approximation for small distances: we will present it in section 6 for particular examples that apply to the calculation of the conductance on graphs.

### 5. General gluing of more than two vertices

The construction is done by recursion: one first glues two vertices using the results of the previous section to get an effective vertex corresponding to this gluing. Then, one glues this effective vertex to a third one. The result is the gluing of the three vertices that we can glue to a fourth one, and so on. The quantum field follows the same rule, and, at the end, we get

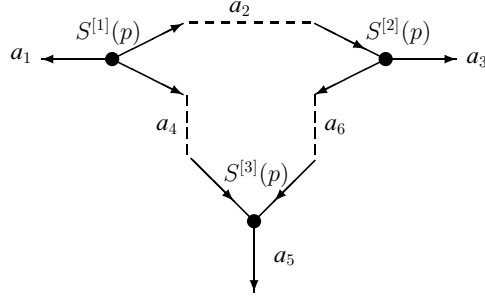


Figure 7. The triangle.

the field for the total graph in terms of the generators  $\{a_a(p), a_a^\dagger(p)\}$  of the *external* edges  $a$  solely. It obeys relations (2.15) on external edges.

The total scattering matrix of a general graph is thus obtained through a recursive use of the formulae of section 4. If we denote by  $S^{[j]}(p)$ ,  $j = 1, \dots, N + 1$ , the local  $S$ -matrices of the  $N + 1$  vertices under consideration, and by  $S^{[j\dots k]}(p)$ ,  $1 \leq j < k \leq N + 1$ , the  $S$ -matrix resulting from the gluing of the vertices  $j$  to  $k$ , we get the recursion formula,

$$S_{\text{tot}}^{[1\dots N+1]}(p) = \begin{pmatrix} S_{\text{tot}}^{[1\dots N+1]}(p)_{11} & S_{\text{tot}}^{[1\dots N+1]}(p)_{12} \\ S_{\text{tot}}^{[1\dots N+1]}(p)_{21} & S_{\text{tot}}^{[1\dots N+1]}(p)_{22} \end{pmatrix} \quad (5.1)$$

$$S_{\text{tot}}^{[1\dots N+1]}(p)_{11} = \tilde{S}_{11}^{[1\dots N]}(p) + \tilde{S}_{12}^{[1\dots N]}(p)D(p)^{-1}S_{11}^{[N+1]}(p)\tilde{S}_{21}^{[1\dots N]}(p) \quad (5.2)$$

$$S_{\text{tot}}^{[1\dots N+1]}(p)_{12} = \tilde{S}_{12}^{[1\dots N]}(p)D(p)^{-1}S_{12}^{[N+1]}(p) \quad (5.3)$$

$$S_{\text{tot}}^{[1\dots N+1]}(p)_{21} = S_{21}^{[N+1]}(p)\tilde{D}(p)^{-1}\tilde{S}_{21}^{[1\dots N]}(p) \quad (5.4)$$

$$S_{\text{tot}}^{[1\dots N+1]}(p)_{22} = S_{22}^{[N+1]}(p) + S_{21}^{[N+1]}(p)\tilde{D}(p)^{-1}\tilde{S}_{22}^{[1\dots N]}(p)S_{12}^{[N+1]}(p) \quad (5.5)$$

$$D(p) = \mathbb{I} - S_{11}^{[N+1]}(p)\tilde{S}_{22}^{[1\dots N]}(p); \quad \tilde{D}(p) = \mathbb{I} - \tilde{S}_{22}^{[1\dots N]}(p)S_{11}^{[N+1]}(p), \quad (5.6)$$

where  $D(p)$  and  $\tilde{D}(p)$  are supposed to be invertible.  $\tilde{S}^{[1\dots N]}(p)$  is deduced from the scattering matrix  $S^{[1\dots N]}(p)$  obtained from the previous step through a reordering of the rows and columns such that the modes ‘glued’ in the step appear at the right place (see section 5.1). Of course, the decomposition of the  $S$ -matrices into submatrices  $S_{11}$ ,  $S_{12}$ ,  $S_{21}$  and  $S_{22}$  and the size of these submatrices depend on the number of edges that are glued between two vertices.

### 5.1. Example 4: star-triangle relation

We consider a graph constituted with  $N = 3$  identical vertices possessing three edges each, coupled as in figure 7. The local boundary conditions are given by

$$\begin{pmatrix} a_1(p) \\ a_4(p) \\ a_2(p) \end{pmatrix} = S^{[1]}(p) \begin{pmatrix} a_1(-p) \\ a_4(-p) \\ a_2(-p) \end{pmatrix}; \quad \begin{pmatrix} a_2(-p) \\ a_3(p) \\ a_6(p) \end{pmatrix} = S^{[2]}(p) \begin{pmatrix} a_2(p) \\ a_3(-p) \\ a_6(-p) \end{pmatrix} \quad (5.7)$$

$$\begin{pmatrix} a_4(-p) \\ a_6(-p) \\ a_5(p) \end{pmatrix} = S^{[3]}(p) \begin{pmatrix} a_4(p) \\ a_6(p) \\ a_5(-p) \end{pmatrix}.$$

We first construct  $S^{[12]}(p)$  as the gluing of  $S^{[1]}(p)$  and  $S^{[2]}(p)$ . Since the vertices are chosen identical, we have

$$S^{[2]}(p) = W_2(p)P_2S^{[1]}(-p)P_2^{-1}W_2(p) = \left( \begin{array}{c|cc} s_{33}(-p)e^{2ipd} & s_{31}(-p)e^{ipd} & s_{32}(-p)e^{ipd} \\ \hline s_{13}(-p)e^{ipd} & s_{11}(-p) & s_{12}(-p) \\ s_{23}(-p)e^{ipd} & s_{21}(-p) & s_{22}(-p) \end{array} \right). \quad (5.8)$$

During this first gluing, the mode  $\alpha_2(p)$  is the only inner mode. Modes  $\alpha_4(p)$  and  $\alpha_6(p)$ , which are inner modes of the full graph, are considered as outer modes for a while. We can apply the results of section 3.2 for the tree graph to get  $S^{[12]}(p)$ : it is in fact of form (4.12), where  $S_{11}(p)$  is the  $2 \times 2$  upper left submatrix of  $S^{[1]}(p)$ .

The inner mode  $\alpha_2(p)$  is constructed from the ‘outer’ modes  $\alpha_a(p)$ ,  $a = 1, 3, 4, 6$ :

$$\alpha_2(p) = \frac{1}{1 - e^{2ipd}s_{33}(p)s_{33}(-p)} \{s_{31}(p)\alpha_1(-p) + s_{32}(p)\alpha_4(-p) + e^{ipd}s_{33}(p)(s_{31}(-p)\alpha_3(-p) + s_{32}(-p)\alpha_6(-p))\}. \quad (5.9)$$

These ‘outer’ modes obey the boundary condition

$$\begin{pmatrix} \alpha_1(p) \\ \alpha_4(p) \\ \alpha_3(p) \\ \alpha_6(p) \end{pmatrix} = S^{[12]}(p) \begin{pmatrix} \alpha_1(-p) \\ \alpha_4(-p) \\ \alpha_3(-p) \\ \alpha_6(-p) \end{pmatrix}. \quad (5.10)$$

We now turn to the second stage of the gluing: we glue  $S^{[12]}(p)$  with  $S^{[3]}(p)$ . Sticking to the identical vertices case, we take  $S^{[3]}(p)$  to be

$$S^{[3]}(p) = W_3(p)P_3S^{[1]}(p)P_3^{-1}W_3(p) = \left( \begin{array}{c|cc} s_{11}(p)e^{2ipd} & s_{13}(p)e^{2ipd} & s_{12}(p)e^{ipd} \\ \hline s_{31}(p)e^{2ipd} & s_{33}(p)e^{2ipd} & s_{32}(p)e^{ipd} \\ s_{21}(p)e^{ipd} & s_{23}(p)e^{ipd} & s_{22}(p) \end{array} \right), \quad (5.11)$$

so that we have a local boundary condition as in (5.7). We have chosen distance  $d$  to be the same on each edge, but clearly nothing changes in the construction if distance  $d_{12}$  between vertices 1 and 2 (appearing at the first stage) is different from distances  $d_{13}$  and  $d_{23}$  (appearing at the second stage). To do the gluing, we have also to reformulate the boundary condition (5.10) in the following way:

$$\begin{pmatrix} \alpha_1(p) \\ \alpha_3(p) \\ \alpha_4(p) \\ \alpha_6(p) \end{pmatrix} = \tilde{S}^{[12]}(p) \begin{pmatrix} \alpha_1(-p) \\ \alpha_3(-p) \\ \alpha_4(-p) \\ \alpha_6(-p) \end{pmatrix}. \quad (5.12)$$

The new  $\tilde{S}^{[12]}(p)$  is deduced from the original  $S^{[12]}(p)$  through the reordering:

$$\tilde{S}^{[12]}(p) = P_{12}S^{[12]}(p)P_{12}^{-1} \quad \text{with} \quad P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.13)$$

Then, one just uses formulae (4.7) with  $\tilde{S}^{[12]}(p)$  playing the role of  $S(p)$ , and  $S^{[3]}(p)$  the role of  $\Sigma(p)$ . In this way, we get a scattering matrix  $S_{\text{tot}}(p)$  for a ‘global’ vertex with three edges, equivalent from ‘outside’ to the three original vertices: this is the star-triangle relation.



Indeed, the boundary condition for outer modes reads

$$\begin{pmatrix} \mathfrak{a}_1(p) \\ \mathfrak{a}_3(p) \\ \mathfrak{a}_5(p) \end{pmatrix} = S_{\text{tot}}(p) \begin{pmatrix} \mathfrak{a}_1(-p) \\ \mathfrak{a}_3(-p) \\ \mathfrak{a}_5(-p) \end{pmatrix} \quad (5.14)$$

and inner modes  $\mathfrak{a}_4(p)$  and  $\mathfrak{a}_6(p)$  are obtained through relation (4.9) with

$$A_2(p) = \begin{pmatrix} \mathfrak{a}_4(p) \\ \mathfrak{a}_6(p) \end{pmatrix}; \quad A_1(p) = \begin{pmatrix} \mathfrak{a}_1(p) \\ \mathfrak{a}_3(p) \end{pmatrix}; \quad A_3(p) = \mathfrak{a}_5(p). \quad (5.15)$$

The complete expression for inner mode  $\mathfrak{a}_2(p)$  is obtained using (5.9) and the expressions for  $\mathfrak{a}_4(p)$  and  $\mathfrak{a}_6(p)$ . However, as the general formulae are rather complicated, we prefer not to write them explicitly. A complete example of the triangle scattering matrix is given in section 6.4 for special (constant) local scattering matrices.

### 6. Scale invariant matrices and Kirchhoff's rule

We focus on the case of identical vertices and apply the formalism to scale invariant matrices. Since we will deal with the examples treated in previous sections, which are constructed from local  $3 \times 3$  scattering matrices, we focus on scale invariant matrices of this size. They have the form

$$S_\alpha = \frac{1}{1 + \alpha_1^2 + \alpha_2^2} \begin{pmatrix} 1 - \alpha_1^2 + \alpha_2^2 & -2\alpha_1\alpha_2 & -2\alpha_1 \\ -2\alpha_1\alpha_2 & 1 + \alpha_1^2 - \alpha_2^2 & -2\alpha_2 \\ -2\alpha_1 & -2\alpha_2 & \alpha_1^2 + \alpha_2^2 - 1 \end{pmatrix}. \quad (6.1)$$

We will see in the following section how to deduce the conductance  $G_{ab}$  between edges  $a$  and  $b$  of a quantum wire from the scattering matrix of this wire. For a star graph and a Luttinger liquid model, the conductance obeys Kirchhoff's rule

$$\sum_{a=1}^n G_{ab} = 0, \quad (6.2)$$

if the scattering matrix obeys [9]

$$\sum_{a=1}^n S_{ab} = 1. \quad (6.3)$$

We will loosely call this relation Kirchhoff's rule (for scattering matrices). For a general quantum wire, one may impose Kirchhoff's rule for scattering matrices *locally*, i.e. on each vertex of the wire, or *globally*, i.e. on the total scattering matrix. To get a scale invariant matrix (6.1) obeying Kirchhoff's rule, one needs to impose the constraint  $1 + \alpha_1 + \alpha_2 = 0$ .

As already mentioned, we now apply the results obtained for the examples 1, 2, 3 and 4 treated in previous sections to cases where the local scattering matrices have the form (6.1).

#### 6.1. Example 1: the tree graph

For the tree graph (see section 3.2) built on local scale invariant matrices, the total scattering matrix (3.23) takes the symmetric form:

$$S_{\text{tot}}(p) = \frac{1}{\mathcal{N}(p)} \begin{pmatrix} S_{11}(p) & S_{12}(p) \\ S_{12}(p) & S_{11}(p) \end{pmatrix} \quad (6.4)$$

$$\mathcal{N}(p) = (1 + \alpha_1^2 + \alpha_2^2)^2 - (1 - \alpha_1^2 - \alpha_2^2)^2 e^{2ipd} \quad (6.5)$$

$$S_{11}(p) = (1 + \alpha_1^2 + \alpha_2^2) \begin{pmatrix} 1 - \alpha_1^2 + \alpha_2^2 & -2\alpha_1\alpha_2 \\ -2\alpha_1\alpha_2 & 1 + \alpha_1^2 - \alpha_2^2 \end{pmatrix} + e^{2ipd} (\alpha_1^2 + \alpha_2^2 - 1) \begin{pmatrix} 1 + \alpha_1^2 - \alpha_2^2 & 2\alpha_1\alpha_2 \\ 2\alpha_1\alpha_2 & 1 - \alpha_1^2 + \alpha_2^2 \end{pmatrix} \quad (6.6)$$

$$S_{12}(p) = 4 e^{ipd} \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{pmatrix}. \quad (6.7)$$

The inner mode is expressed as

$$\mathbf{a}_3(p) = \frac{-2}{\mathcal{N}(p)} \{ (\alpha_1^2 + \alpha_2^2 + 1) (\alpha_1 \mathbf{a}_1(-p) + \alpha_2 \mathbf{a}_2(-p)) + (\alpha_1^2 + \alpha_2^2 - 1) e^{ipd} (\alpha_1 \mathbf{a}_4(-p) + \alpha_2 \mathbf{a}_5(-p)) \}, \quad (6.8)$$

where, for the edges, we have used the numbering given in figure 3.

When one considers the particular case  $\alpha_1 = \alpha_2 = \pm 1$ , one recovers the scattering matrix computed in example (IV.4) of [6].

*6.1.1. Approximation for short distances.* We first rewrite the scattering matrix as

$$S_{\text{tot}}(p) = \frac{4}{N(p)} \begin{pmatrix} (1 - t^2)A_0 + itA_1 & B \\ B & (1 - t^2)A_0 + itA_1 \end{pmatrix} \quad (6.9)$$

$$N(p) = (\mu^2 - 1)t^2 - 2i(1 + \mu^2)t + 1 - \mu^2 \quad \text{with} \quad \mu = \frac{1 - \alpha_1^2 - \alpha_2^2}{1 + \alpha_1^2 + \alpha_2^2} \quad (6.10)$$

$$A_0 = \frac{1}{(1 + \alpha_1^2 + \alpha_2^2)^2} \begin{pmatrix} \alpha_2^2 & -\alpha_1\alpha_2 \\ -\alpha_1\alpha_2 & \alpha_1^2 \end{pmatrix} \quad (6.11)$$

$$A_1 = \frac{1}{(1 + \alpha_1^2 + \alpha_2^2)^2} \begin{pmatrix} \alpha_1^4 - \alpha_2^4 - 1 & \alpha_1\alpha_2(\alpha_1^2 + \alpha_2^2) \\ \alpha_1\alpha_2(\alpha_1^2 + \alpha_2^2) & \alpha_2^4 - \alpha_1^4 - 1 \end{pmatrix} \quad (6.12)$$

$$B = \frac{1}{(1 + \alpha_1^2 + \alpha_2^2)^2} \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{pmatrix}, \quad (6.13)$$

where  $t = \tan(\frac{dp}{2})$ . Taking  $d = 0$ , we get a new  $4 \times 4$  scattering matrix

$$S^{(0)} = \frac{4}{1 - \mu^2} \begin{pmatrix} A_0 & B \\ B & A_0 \end{pmatrix} = \frac{1}{\alpha_1^2 + \alpha_2^2} \begin{pmatrix} \alpha_2^2 & -\alpha_1\alpha_2 & \alpha_1^2 & \alpha_1\alpha_2 \\ -\alpha_1\alpha_2 & \alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2 \\ \alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2 & -\alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 & -\alpha_1\alpha_2 & \alpha_1^2 \end{pmatrix}. \quad (6.14)$$

Remark that this matrix obeys Kirchhoff's rule, whatever the values of  $\alpha_1$  and  $\alpha_2$  are, even when the local scattering matrices do not. For<sup>2</sup>  $\alpha_2 = \epsilon\alpha_1$ , with  $\epsilon = \pm 1$ , we get simpler matrices (still obeying Kirchhoff's rule)

$$S^{(0)} = \frac{1}{2} \begin{pmatrix} 1 & -\epsilon & 1 & \epsilon \\ -\epsilon & 1 & \epsilon & 1 \\ 1 & \epsilon & 1 & -\epsilon \\ \epsilon & 1 & -\epsilon & 1 \end{pmatrix} \quad \text{with} \quad \epsilon = \pm 1. \quad (6.15)$$

<sup>2</sup> Note that if one also imposes Kirchhoff's rule on local vertices, one needs to take  $\epsilon = +1$ .

These matrices can be compared with the two matrices introduced in [14] for the modelization of a condensed matter experiment proposal:

$$S_{ch}^{(0)} = \frac{1}{2} \begin{pmatrix} \epsilon & 1 & -\epsilon & 1 \\ 1 & \epsilon & 1 & -\epsilon \\ -\epsilon & 1 & \epsilon & 1 \\ 1 & -\epsilon & 1 & \epsilon \end{pmatrix} \quad \text{with } \epsilon = \pm 1. \quad (6.16)$$

Indeed, one has

$$S_{ch}^{(0)} = U S^{(0)} \quad \text{where } U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (6.17)$$

Since  $U S^{(0)} U = S^{(0)}$ , the unitarity relations are preserved by this transformation, and indeed  $S^{(0)}$  and  $S_{ch}^{(0)}$  are unitary.

The ‘first-order’ correction (in terms of  $d$ , the distance between the two vertices) can be computed as in section 3, setting  $t \sim \frac{dp}{2}$  in the above matrices. We focus on the case  $\alpha_2 = \epsilon \alpha_1$  and set  $\beta = \alpha_1^2$ . Multiplying by  $U$ , we get a first correction (in  $d$ ) for the matrices given in [14]:

$$S^{(1)}(p) \sim \frac{-1/2}{(pd + 4i\beta)(pd + i/\beta)} \left\{ 8S_{ch}^{(0)} + i\frac{dp}{\beta} S_{ch}^{(1)} - (dp)^2 S_{ch}^{(2)} \right\}, \quad (6.18)$$

$$S_{ch}^{(1)} = \begin{pmatrix} 0 & 0 & 4\epsilon\beta^2 & -1 \\ 0 & 0 & -1 & 4\epsilon\beta^2 \\ 4\epsilon\beta^2 & -1 & 0 & 0 \\ -1 & 4\epsilon\beta^2 & 0 & 0 \end{pmatrix}; \quad S_{ch}^{(2)} = \begin{pmatrix} 0 & 0 & -\epsilon & 1 \\ 0 & 0 & 1 & -\epsilon \\ -\epsilon & 1 & 0 & 0 \\ 1 & -\epsilon & 0 & 0 \end{pmatrix}. \quad (6.19)$$

## 6.2. Example 2: the loop

The loop graph has been treated in section 4.4. Using the form (6.1), the total scattering matrix (4.25) rewrites

$$S_{\text{tot}}(p) = \frac{\exp(idp)}{\mathcal{N}(p)} \begin{pmatrix} -2i\mu \sin(dp) & 1 - \mu^2 \\ 1 - \mu^2 & -2i\mu \sin(dp) \end{pmatrix} \quad \text{with } \mu = \frac{1 - \alpha_1^2 + \alpha_2^2}{1 + \alpha_1^2 + \alpha_2^2} \\ \mathcal{N}(p) = 1 - (\mu \exp(idp))^2. \quad (6.20)$$

For scattering matrices obeying locally Kirchhoff’s rule, one has  $\mu = \frac{1+\alpha_1}{1+\alpha_1+\alpha_1^2}$ .

In the particular case  $\mu = -\frac{1}{3}$  (i.e.,  $\alpha_1 = -2$  when Kirchhoff’s rule is obeyed locally), one recovers the  $S$  matrix found in example 3.2 of [4], with identification  $p \equiv \sqrt{E}$ .

The inner modes take the form

$$\mathbf{a}_2(p) = \frac{\gamma \alpha_2}{\mathcal{N}(p)} (e^{-ipd} \mathbf{a}_1(-p) + \mu \mathbf{a}_3(-p)) \\ \mathbf{a}_4(p) = \frac{\gamma}{\mathcal{N}(p)} (e^{-ipd} \mathbf{a}_1(-p) + \mu \mathbf{a}_3(-p)) \quad \gamma = \frac{2\alpha_1}{1 + \alpha_1^2 + \alpha_2^2}. \quad (6.21)$$

6.2.1. *Expansion in terms of the loop length.* We rewrite the scattering matrix in terms of  $t = \tan(dp/2)$ :

$$S_{\text{tot}}(p) = \begin{pmatrix} R(p) & T(p) \\ T(p) & R(p) \end{pmatrix} \quad \text{with} \quad \begin{cases} R(p) = \frac{4i\mu t}{(1 - \mu^2)t^2 + 2i(1 + \mu^2)t - 1 + \mu^2} \\ T(p) = \frac{(\mu^2 - 1)(1 + t^2)}{(1 - \mu^2)t^2 + 2i(1 + \mu^2)t - 1 + \mu^2} \end{cases}. \quad (6.22)$$

When  $d \rightarrow 0$ , we get an approximation of the scattering matrix setting  $t \sim dp/2$ . One recognizes in the approximation the scattering matrix for a point-like impurity on the line. The reflection and transmission coefficients defining this impurity are given by local parameters  $\alpha_1, \alpha_2$ , and by distance  $d$  (or equivalently by the surface  $d^2$  of the loop). Correction to this approximation, induced by the surface of the loop, is given by the full expression (6.20).

### 6.3. Example 3: the tadpole

We apply the result for the tadpole graph (see section 4.5). Starting from the general form (6.1) and using the expression (4.32), this leads to the tadpole  $S$  matrix

$$S_{\text{tot}}(p) = \begin{pmatrix} R(p) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad R(p) = \frac{(1 + \alpha_1^2 + \alpha_2^2) e^{2idp} + 4\alpha_2 e^{idp} + 1 - \alpha_1^2 + \alpha_2^2}{(1 - \alpha_1^2 + \alpha_2^2) e^{2idp} + 4\alpha_2 e^{idp} + 1 + \alpha_1^2 + \alpha_2^2}. \quad (6.23)$$

We get a system with a half-line with reflection coefficient 1 decoupled from another half-line, with reflection coefficient  $R(p)$ .

In the particular case of  $\alpha_1 = \pm 2$  and  $\alpha_2 = 1$  one recovers the  $S$  matrix given in example 4.3 of [4], again with identification  $p \equiv \sqrt{E}$ .

The modes inside the loop read

$$a_2(p) = \frac{-2\alpha_1(\alpha_2 + e^{ipd})}{(1 - \alpha_1^2 + \alpha_2^2) e^{2idp} + 4\alpha_2 e^{idp} + 1 + \alpha_1^2 + \alpha_2^2} a_1(-p) \quad (6.24)$$

$$a_4(p) = \frac{-2\alpha_1(1 + \alpha_2 e^{ipd})}{(1 - \alpha_1^2 + \alpha_2^2) e^{2idp} + 4\alpha_2 e^{idp} + 1 + \alpha_1^2 + \alpha_2^2} a_1(-p). \quad (6.25)$$

6.3.1. *Expansion in terms of the loop length.* We rewrite the reflection coefficient (6.23) as

$$R(p) = \frac{(1 - \alpha_2)^2 t^2 - 2i\alpha_1^2 t - (1 + \alpha_2)^2}{(1 - \alpha_2)^2 t^2 + 2i\alpha_1^2 t - (1 + \alpha_2)^2} \quad \text{with} \quad t = \tan(pd/2) \quad (6.26)$$

and perform the short distance approximation setting  $t \sim dp/2$ .

### 6.4. Example 4: the triangle

We present the calculation of the total scattering matrix for the triangle (as has been explained in section 5.1), up to the end, for scale invariant local scattering matrices (6.1).

To simplify the presentation, we consider three identical vertices with local scattering matrix (6.1) with  $\alpha_1 = \alpha_2 = 1$ ,

$$S_0 = -\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix},$$

and take the same length  $d$  for the three connecting lines.

At the first step of the gluing, we get a tree-graph matrix of type (6.7),

$$S^{[12]}(p) = \frac{1}{9 - e^{2idp}} \begin{pmatrix} e^{2idp} + 3 & 2(e^{2idp} - 3) & 4e^{idp} & 4e^{idp} \\ 2(e^{2idp} - 3) & e^{2idp} + 3 & 4e^{idp} & 4e^{idp} \\ 4e^{idp} & 4e^{idp} & e^{2idp} + 3 & 2(e^{2idp} - 3) \\ 4e^{idp} & 4e^{idp} & 2(e^{2idp} - 3) & e^{2idp} + 3 \end{pmatrix}$$

that, after rotation by  $P_{12}$ , we glue to

$$S^{[3]}(p) = -\frac{1}{3} \begin{pmatrix} -e^{2idp} & 2e^{2idp} & 2e^{idp} \\ 2e^{2idp} & -e^{2idp} & 2e^{idp} \\ 2e^{idp} & 2e^{idp} & -1 \end{pmatrix}$$

using the general formulae (4.7). We get

$$S_{\text{tot}}(p) = \frac{3e^{ipd} + 1}{e^{ipd} + 3} \mathbb{I}_3 + \frac{4(e^{ipd} - 1)e^{ipd}}{(e^{ipd} + 3)(e^{2ipd} - 2e^{ipd} + 3)} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (6.27)$$

Inner modes  $\mathbf{a}_4(p)$  and  $\mathbf{a}_6(p)$  are obtained using formulae (4.9):

$$\mathbf{a}_4(p) = \frac{2}{e^{2ipd} - 2e^{ipd} + 3} \left\{ \frac{e^{ipd} - 3}{e^{ipd} + 3} \mathbf{a}_1(-p) + \frac{2e^{ipd}}{e^{ipd} + 3} \mathbf{a}_3(-p) - \frac{e^{ipd}(e^{ipd} + 1)}{e^{ipd} + 3} \mathbf{a}_5(-p) \right\}$$

$$\mathbf{a}_6(p) = \frac{2}{e^{2ipd} - 2e^{ipd} + 3} \left\{ \frac{2e^{ipd}}{e^{ipd} + 3} \mathbf{a}_1(-p) + \frac{e^{ipd} - 3}{e^{ipd} + 3} \mathbf{a}_3(-p) - \frac{e^{ipd}(e^{ipd} + 1)}{e^{ipd} + 3} \mathbf{a}_5(-p) \right\}.$$

Mode  $\mathbf{a}_2(p)$  is obtained according to the calculation explained in section 5.1:

$$\mathbf{a}_2(p) = \frac{2}{e^{2idp} - 9} \{3\mathbf{a}_1(-p) + 3\mathbf{a}_4(-p) + e^{ipd} \mathbf{a}_6(p) + e^{ipd} \mathbf{a}_3(-p)\}.$$

6.4.1. *Expansion in terms of the distance.* An equivalent form of (6.27) is given by

$$S_{\text{tot}}(p) = -\frac{t - 2i}{t + 2i} \mathbb{I}_3 + \frac{2t(t^2 + 1)}{(t + 2i)(3t^2 + 2it - 1)} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad (6.28)$$

where  $t = \tan(dp/2)$ , with the short distance expansion given by  $t \sim dp/2$ .

One can compare this matrix with the symmetric scattering matrix introduced in [13]:

$$\mathcal{R} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \quad (6.29)$$

To get this matrix at  $d = 0$ , we multiply  $S_{\text{tot}}(p)$  by  $\mathcal{R}$ . It is possible because we have  $\mathcal{R}^2 = \mathbb{I}$  and  $[S_{\text{tot}}(p), \mathcal{R}] = 0$ , so that  $\mathcal{R}S_{\text{tot}}(p)$  is still unitary. Then, the short distance approximation leads to

$$\mathcal{R}S_{\text{tot}}(p) \sim -\frac{t - 2i}{t + 2i} \mathcal{R} + \frac{2t(t^2 + 1)}{(t + 2i)(3t^2 + 2it - 1)} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad t \sim \frac{dp}{2}, \quad (6.30)$$

which can be viewed as a first correction to the scattering matrix  $\mathcal{R}$ .

## 7. Conductance

### 7.1. General settings

The conductance of a quantum wire is obtained through the linear response of the current  $j_{\mu,a}(x, t)$  to a classical external potential  $A_{\mu,a}(x, t)$  minimally coupled to a fermionic field  $\psi_b(x, t)$  on the quantum wire. This has been treated in [10] for the Tomonaga–Luttinger (TL) model in the case of star graphs; see also [7] for a general treatment of a self-adjoint magnetic Laplacian on graphs. When considering a general quantum wire, we will restrict ourselves to the problem of computing the conductance within the TL model and between external edges only. Via bosonization, all of the problem can be rewritten in terms of the bosonic field  $\phi_a(x, t)$  given in (2.14); see [10] for details. For instance, the local gauge transformations read

$$A_{\mu,a}(x, t) \rightarrow A_{\mu,a}(x, t) - \partial_\mu \Lambda_a(x, t), \quad \mu = x, t; \quad a = 1, \dots, n \quad (7.1)$$

$$\phi_a(x, t) \rightarrow \phi_a(x, t) + \frac{1}{\sigma\sqrt{\pi}} \Lambda_a(x, t) \quad (7.2)$$

and the corresponding invariant current reads

$$j_{\mu,a}(x, t) = \sqrt{\pi} \partial_\mu \phi_a(x, t) + \frac{1}{\sigma} A_{\mu,a}(x, t), \quad (7.3)$$

with possibly some additional terms corresponding to bound states [11]. Let us stress that this current is just the bosonized version of a fermionic (relativistic) current  $\psi(x, t) \gamma_\mu \psi(x, t)$ . Then, the linear response theory leads to

$$\langle j_{x,a}(x, t) \rangle_{A_\mu} = \frac{1}{\sigma} A_{x,a}(x, t) + \frac{i}{\sigma} \sum_{b=1}^n \int_{-\infty}^t d\tau \int_0^\infty dy A_{x,b}(y, \tau) [\partial_x \phi_b(y, \tau), \partial_x \phi_a(x, t)].$$

Considering a uniform electric field switched on at  $t = t_0$ , in the Weyl gauge,

$$E_a(t) = \partial_t A_{x,a}(t) \quad \text{with} \quad A_{x,a}(t) = 0 \quad \text{if} \quad t < t_0 \quad (7.4)$$

$$A_{t,a} = 0, \quad \forall t, \quad a = 1, \dots, n, \quad (7.5)$$

and supposing that the scattering matrix is symmetric and admits simple non-real poles only, one can derive the conductance [11]

$$\langle j_{x,a}(x, t) \rangle_{A_\mu} = \sum_{b=1}^n \int_{-\infty}^\infty \frac{d\omega}{2\pi} \widehat{A}_{x,b}(\omega) G_{ab}(\omega, t - t_0) e^{-i\omega t}, \quad (7.6)$$

$$G_{ab}(\omega, t) = G_0 \left\{ \delta_{ab} - S_{ab}(\omega) - \sum_{\eta \in \mathcal{P}} \frac{e^{i(\omega - i\eta)t}}{\omega - i\eta} \mathcal{T}_{ab}(\eta) \right\}. \quad (7.7)$$

We have also introduced  $\mathcal{P}$ , the set of poles of the scattering matrix and

$$\mathcal{T}_{ab}(\eta) = \lim_{p \rightarrow i\eta} (p - i\eta) S_{ab}(p). \quad (7.8)$$

$G_0$  is the conductance for the infinite line. The conductance depends on the time  $t_0$  of the switch-on of the electric field, but also (due to the presence of poles in the scattering matrix) on its frequency  $\omega$ .

*Short distance approximation for the conductance.* The formula (7.7) can be applied to any of the scattering matrices computed in the previous sections, in particular to those deduced

in the short distance approximation, where the number of poles is finite. Moreover, when performing this short distance approximation, and since the pole content has already been (partially) kept in the  $\mathcal{T}_{ab}$  matrix, one can take for the total scattering matrix in (7.7) its value for  $d = 0$ . Since in our examples the local scattering matrices are constant, this is equivalent to taking  $p = 0$  in  $S$ , so that we get for the conductance the approximated form

$$G_{ab}^{\text{approx}}(\omega, t) = G_0 \left\{ \delta_{ab} - S_{ab}(0) - \sum_{\eta \in \mathcal{P}_0} \frac{e^{i(\omega - i\eta)t}}{\omega - i\eta} \mathcal{T}_{ab}(\eta) \right\}, \quad (7.9)$$

where  $\mathcal{P}_0$  is the set of poles appearing in the approximated scattering matrix of the graph under consideration. An obvious refinement of this approximation is to take

$$G_{ab}^{\text{refin}}(\omega, t) = G_0 \left\{ \delta_{ab} - S_{ab}^{\text{approx}}(\omega) - \sum_{\eta \in \mathcal{P}_0} \frac{e^{i(\omega - i\eta)t}}{\omega - i\eta} \mathcal{T}_{ab}(\eta) \right\}, \quad (7.10)$$

where  $S_{ab}^{\text{approx}}(\omega)$  is the short distance approximation of the scattering matrix.

Some examples of such calculations are done in the following section.

### 7.2. Examples

We apply the above formalism to the examples dealt with in section 6. Except for one particular case, we will consider the scattering matrix within the short distance approximation, as has been presented in section 4.3. In the case of the tadpole, we perform the exact calculation and show that the short distance approximation is in accordance with the exact result, justifying in this way the approximation.

**7.2.1. Tree graph.** We start with matrix (6.18), which possesses two simple poles

$$i\eta_1 = \frac{-i}{d}4\beta \quad \text{and} \quad i\eta_2 = \frac{i}{d\beta}. \quad (7.11)$$

We recall that  $\beta = \alpha_1^2$ . One gets, using notations (6.19),

$$\mathcal{T}_1 = \mathcal{T}(\eta_1) = \frac{-2i\beta}{4\beta^2 - 1} \left\{ 2S_{ch}^{(0)} + S_{ch}^{(1)} + 4\beta^2 S_{ch}^{(2)} \right\} \quad (7.12)$$

$$\mathcal{T}_2 = \mathcal{T}(\eta_2) = \frac{i\beta}{4\beta^2 - 1} \left\{ 4S_{ch}^{(0)} + \frac{1}{2}S_{ch}^{(1)} + \frac{1}{2\beta^2}S_{ch}^{(2)} \right\}. \quad (7.13)$$

Starting from the formula (7.9), it leads to a conductance

$$G(\omega, t) = G_0 \left\{ \frac{1}{2} \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} - \frac{e^{i(\omega - i\eta_1)t}}{\omega - i\eta_1} \mathcal{T}_1 - \frac{e^{i(\omega - i\eta_2)t}}{\omega - i\eta_2} \mathcal{T}_2 \right\}, \quad (7.14)$$

where we have introduced the  $2 \times 2$  matrix

$$A = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}. \quad (7.15)$$

**7.2.2. Loop.** We consider the loop scattering matrix (6.22). It possesses two simple poles

$$i\eta_1 = \frac{2i(\mu - 1)}{d(\mu + 1)} = \frac{-2i\alpha_1^2}{(1 + \alpha_2^2)d} \quad \text{and} \quad i\eta_2 = \frac{2i(\mu + 1)}{d(\mu - 1)} = \frac{-2i(1 + \alpha_2^2)}{\alpha_1^2 d}. \quad (7.16)$$

They lead to the two matrices

$$\mathcal{T}_1 = \mathcal{T}(\eta_1) = i \frac{1+\mu}{1-\mu} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{T}_2 = \mathcal{T}(\eta_2) = -i \frac{1-\mu}{1+\mu} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (7.17)$$

Then, the conductance (7.9) is rewritten as

$$G(\omega, t) = G_0 \left\{ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \frac{e^{i(\omega-i\eta_1)t}}{\omega - i\eta_1} \mathcal{T}_1 - \frac{e^{i(\omega-i\eta_2)t}}{\omega - i\eta_2} \mathcal{T}_2 \right\}. \quad (7.18)$$

**7.2.3. Tadpole.** To simplify the presentation, we consider the case  $\alpha_2 = 1$ , but the same sort of calculation can be done for the general case.

*Exact calculation.* The expression (6.23) for  $\alpha_2 = 1$  simplifies to

$$R(p) = \frac{2 + i\alpha_1^2 t}{2 - i\alpha_1^2 t}. \quad (7.19)$$

The poles of  $R(p)$  are given by

$$\tan\left(\frac{dp}{2}\right) = \frac{-2i}{\alpha_1^2} \quad \Leftrightarrow \quad p = i\eta_k = i\eta_0 + \frac{2k\pi}{d},$$

$$k \in \mathbb{Z} \quad \text{with} \quad \eta_0 = -\frac{2}{d} \operatorname{arctanh}\left(\frac{2}{\alpha_1^2}\right),$$

which leads to

$$\mathcal{T}_k = \mathcal{T}(\eta_k) = \frac{8i}{d} \frac{\alpha_1^2}{\alpha_1^4 - 4}, \quad \forall k \in \mathbb{Z}. \quad (7.20)$$

Hence, we get

$$G(\omega, t) = G_0 \left\{ 1 - R(\omega) - \frac{8i}{d} \frac{\alpha_1^2}{\alpha_1^4 - 4} \sum_{k \in \mathbb{Z}} \frac{e^{i(\omega-i\eta_k)t}}{\omega - i\eta_k} \right\}. \quad (7.21)$$

The sum can be computed for real parameter  $\alpha_1$ , and one obtains

$$\sum_{k \in \mathbb{Z}} \frac{1}{\omega - i\eta_k} e^{i(\omega-i\eta_k)t} = \frac{-id e^{in_0(t)d(\omega-i\eta_0)}}{1 - e^{id(\omega-i\eta_0)}} \quad \text{with} \quad n_0(t) = \left[ \frac{t}{d} \right], \quad (7.22)$$

where  $[\cdot]$  denotes the integer part. This leads to

$$G(\omega, t) = G_0 \left\{ 1 - \frac{2 + i\alpha_1^2 \tan\left(\frac{\omega d}{2}\right)}{2 - i\alpha_1^2 \tan\left(\frac{\omega d}{2}\right)} - \frac{8\alpha_1^2}{\alpha_1^4 - 4} \frac{e^{in_0(t)d(\omega-i\eta_0)}}{1 - e^{id(\omega-i\eta_0)}} \right\}. \quad (7.23)$$

*Short distance approximation.* If one performs the same calculation with approximation  $t \sim pd/2$ , we get a single simple pole  $i\eta' = -4i/d\alpha_1^2$  that leads to

$$G_{\text{approx}}(\omega, t) = G_0 \left\{ 1 - \frac{2 + i\alpha_1^2 \frac{\omega d}{2}}{2 - i\alpha_1^2 \frac{\omega d}{2}} + \frac{8i}{d\alpha_1^2} \frac{e^{i(\omega-i\eta')t}}{\omega - i\eta'} \right\}. \quad (7.24)$$

To compare this latter expression with the exact result, we first note that

$$\eta_0 = -\frac{2}{d} \operatorname{arctanh}\left(\frac{2}{\alpha_1^2}\right) \sim -\frac{4}{d\alpha_1^2} = \eta' \quad \text{for} \quad \alpha_1^2 \gg 2. \quad (7.25)$$



Taking this regime for parameter  $\alpha_1$ , we perform an expansion in  $d$  of (7.23), remarking that  $n_0(t)d \sim t$  when  $d \rightarrow 0$ ,

$$G(\omega, t) \sim G_0 \left\{ 1 - \frac{2 + i\alpha_1^2 \frac{\omega d}{2}}{2 - i\alpha_1^2 \frac{\omega d}{2}} + \frac{8}{\alpha_1^2} \frac{e^{i(\omega - i\eta')t}}{(-id)(\omega - i\eta')} \right\}, \quad (7.26)$$

which is exactly the expression of  $G_{\text{approx}}(\omega, t)$ . Thus, the short distance approximation gives a correct estimate of the conductance for this parameter range.

**7.2.4. Triangle.** The matrix (6.30) possesses three simple poles

$$i\eta_0 = \frac{-4i}{d}; \quad i\eta_{\pm} = \frac{-2i \pm 2\sqrt{2}}{3d}, \quad (7.27)$$

leading to

$$T(\eta_0) = \frac{8i}{3d} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad T(\eta_{\pm}) = -\frac{2}{9d} (2i \pm \sqrt{2}) \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (7.28)$$

The conductance takes the form

$$G(\omega, t) = G_0 \left\{ \mathbb{1}_3 - \mathcal{R} - \frac{e^{i(\omega - i\eta_0)t}}{\omega - i\eta_0} T(\eta_0) - \frac{e^{i(\omega - i\eta_+)t}}{\omega - i\eta_+} T(\eta_+) - \frac{e^{i(\omega - i\eta_-)t}}{\omega - i\eta_-} T(\eta_-) \right\}. \quad (7.29)$$

When the distance  $d \rightarrow 0$ , the  $\omega$  dependent part of the conductance goes to zero, and one recovers the conductance computed in [14]. For non-vanishing values of  $d$ , we get  $\omega$  dependent corrections to this conductance.

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## Appendix. Proofs

### A.1. Compatibility relation

We start with the relations at each vertex

$$A_1(p) = S_{11}(p)A_1(-p) + S_{12}(p)A_2(-p) \quad (A.1)$$

$$A_2(p) = S_{21}(p)A_1(-p) + S_{22}(p)A_2(-p) \quad (A.2)$$

$$A_2(-p) = \Sigma_{11}(p)A_2(p) + \Sigma_{12}(p)A_3(-p) \quad (A.3)$$

$$A_3(p) = \Sigma_{21}(p)A_2(p) + \Sigma_{22}(p)A_3(-p), \quad (A.4)$$

where we used the notations of section 4.1. Equations (A.2) and (A.3) allow us to express  $A_2(p)$  in terms of  $A_1(p)$  and  $A_3(p)$  in two different ways:

$$A_2(p) = \tilde{D}(p)^{-1} (S_{21}(p)A_1(-p) + S_{22}(p)\Sigma_{12}(p)A_3(-p)) \quad (A.5)$$

$$A_2(-p) = D(p)^{-1} (\Sigma_{12}(p)A_3(-p) + \Sigma_{11}(p)S_{21}(p)A_1(-p)) \quad (A.6)$$

$$\text{with } \tilde{D}(p) = \mathbb{I} - S_{22}(p)\Sigma_{11}(p) \quad \text{and} \quad D(p) = \mathbb{I} - \Sigma_{11}(p)S_{22}(p). \quad (\text{A.7})$$

Plugging (A.5) into (A.4) and (A.6) into (A.1) leads to relations (4.6)–(4.7). (A.5) can be viewed as the determination of  $A_2(p)$  in terms of  $A_1(p)$  and  $A_3(p)$ . It remains a compatibility relation,

$$\begin{aligned} \tilde{D}(p)^{-1}(S_{21}(p)A_1(-p) + S_{22}(p)\Sigma_{12}(p)A_3(-p)) \\ = D(-p)^{-1}(\Sigma_{12}(-p)A_3(p) + \Sigma_{11}(-p)S_{21}(-p)A_1(p)), \end{aligned} \quad (\text{A.8})$$

which rewrites, using again relations (4.6)–(4.7),

$$\begin{aligned} D(-p)\tilde{D}(p)^{-1}(S_{21}(p)A_1(-p) + S_{22}(p)\Sigma_{12}(p)A_3(-p)) &= (\Sigma_{12}(-p)\Sigma_{21}(p)\tilde{D}(p)^{-1} \\ &\quad - \Sigma_{11}(-p)S_{22}(-p) + \Sigma_{11}(-p)S_{21}(-p)S_{12}(p)D(p)^{-1} \\ &\quad \times \Sigma_{11}(-p))S_{21}(p)A_1(-p) + (\Sigma_{11}(-p)S_{21}(-p)S_{12}(p)D(p)^{-1} - \Sigma_{11}(-p) \\ &\quad + \Sigma_{12}(-p)\Sigma_{21}(p)\tilde{D}(p)^{-1}S_{22}(p))\Sigma_{12}(p)A_3(-p). \end{aligned}$$

Instead of proving this relation, we prove the two following relations that obviously imply the compatibility relation:

$$\begin{aligned} D(-p)\tilde{D}(p)^{-1} &= \Sigma_{12}(-p)\Sigma_{21}(p)\tilde{D}(p)^{-1} \\ &\quad + \Sigma_{11}(-p)S_{21}(-p)S_{12}(p)D(p)^{-1}\Sigma_{11}(p) - \Sigma_{11}(-p)S_{22}(-p) \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} D(-p)\tilde{D}(p)^{-1}S_{22}(p) &= \Sigma_{11}(-p)S_{21}(-p)S_{12}(p)D(p)^{-1} \\ &\quad + \Sigma_{12}(-p)\Sigma_{21}(p)\tilde{D}(p)^{-1}S_{22}(p) - \Sigma_{11}(-p). \end{aligned} \quad (\text{A.10})$$

We start by proving relation (A.9). Multiplying on the right by  $\tilde{D}(p)$  and using the consistency relation (2.8) for  $S(p)$  and  $\Sigma(p)$ , it can be rewritten as

$$\begin{aligned} D(-p) &= \mathbb{I} - \Sigma_{11}(-p)\Sigma_{11}(p) + \Sigma_{11}(-p)(\mathbb{I} - S_{22}(-p)S_{22}(p))D(p)^{-1}\Sigma_{11}(p)\tilde{D}(p) \\ &\quad - \Sigma_{11}(-p)S_{22}(-p)(\mathbb{I} - S_{22}(p)\Sigma_{11}(p)) \end{aligned} \quad (\text{A.11})$$

that is indeed an equality. Relation (A.10) is equivalent to relation (A.9) multiplied from the right by  $S_{22}(p)$ .

## A.2. Consistency and Hermitian analyticity relations

We prove that the scattering matrix (4.7) obeys the consistency and Hermitian analyticity relations (2.8) and (2.9) as soon as the local scattering matrices do. The proof relies on the relations (proven by the direct calculation):

$$\Sigma_{11}(p)\tilde{D}(p)^{-1} = D(p)^{-1}\Sigma_{11}(p) \quad \text{and} \quad \tilde{D}(p)^{-1}S_{22}(p) = S_{22}(p)D(p)^{-1} \quad (\text{A.12})$$

$$(\tilde{D}(p))^\dagger = D(-p) \quad \text{and} \quad (D(p))^\dagger = \tilde{D}(-p). \quad (\text{A.13})$$

Thanks to these relations, it is easy to show that  $S_{\text{tot}}(p)$  is Hermitian analytical. For instance one has

$$\begin{aligned} (S_{\text{tot}}(p))_{11}^\dagger &= S_{11}(p)^\dagger + S_{21}(p)^\dagger\Sigma_{11}(p)^\dagger(D(p)^{-1})^\dagger S_{12}(p)^\dagger \\ &= S_{11}(-p) + S_{21}(-p)\Sigma_{11}(-p)\tilde{D}(-p)^{-1}S_{12}(-p) = (S_{\text{tot}}(-p))_{11}. \end{aligned}$$

The proof of the consistency relation requires more calculation. Considering the 11 component and using consistency relations for  $S(p)$  and  $\Sigma(p)$ , one can rewrite it as

$$\begin{aligned} (S_{\text{tot}}(p)S_{\text{tot}}(-p))_{11} &= S_{11}(p)S_{11}(-p) + S_{12}(p)D(p)^{-1}\{\dots\}S_{21}(-p) \\ \{\dots\} &= D(p) + [\dots]\tilde{D}(-p)^{-1} \\ [\dots] &= (\Sigma_{11}(p) - S_{22}(-p))D(-p)^{-1}(\dots) \\ (\dots) &= -D(-p)\Sigma_{11}(-p) + \Sigma_{11}(-p)\tilde{D}(-p) = 0, \end{aligned} \tag{A.14}$$

where in the last step we used (A.12).

The other relations are proven along the same lines.

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